Planar Topology: Lecture 1 - Point-Set Topology

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1 Basic Definitions

Topological spaces form the study of *generalized geometry* where things may be stretched and squeezed and still thought of as the *same* object. The definition comes from the study of metric spaces.

Definition 1.1 (Metric Space) A metric space is an ordered pair $(X,\, d)$ where X is a set and $d : X \times X \to \mathbb{R}$ is a function such that for all $x,\, y,\, z \in X$:

$$\begin{aligned} &d(x,\, y) \geq 0 && \text{(Positivity)} \\ &d(x,\, x) = 0 && \text{(Definiteness)} \\ &d(x,\, y) = d(y,\, x) && \text{(Symmetry)} \\ &d(x,\, z) \leq d(x,\, y) + d(y,\, z) && \text{(Triangle Inequality)} \end{aligned}$$

The function d is called a *metric* on X. The elements of X are usually referred to as *points*. ■

Example 1.1 The quintessential example is the standard metric on the real line. Equip $\mathbb{R}$ with the function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by:

$$d(x,\, y) = |x - y| \tag{1}$$

That is, d is defined by the absolute value function. From real analysis we know the absolute value function satisfies the triangle inequality (the proof is not hard, either). Positive-definiteness and symmetry are almost immediate from the definition as well. This is the *distance* between two real numbers on the real line. ■

Example 1.2 The Pythagoras theorem gives us a *distance* formula on $\mathbb{R}^N$. Given two points $\mathbf{x},\, \mathbf{y} \in \mathbb{R}^N$ we may define:

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{n=0}^{N-1} (x_n - y_n)^2} \tag{2}$$

where x_n and y_n are the n^{th} components of $\mathbf{x}$ and $\mathbf{y}$, respectively. This is the *Euclidean* metric on $\mathbb{R}^N$, also called the *standard* metric. ■

Example 1.3 You can place different metrics on the same set. The *Manhattan* metric on $\mathbb{R}^N$ is defined by:

$$d(\mathbf{x},\, \mathbf{y}) = \sum_{n=0}^{N-1} |x_n - y_n| \tag{3}$$

That this is a metric can be proved by induction. The base case $N = 1$ is the standard metric on $\mathbb{R}$. ■

Example 1.4 A *norm* on a (real or complex) vector space V is a function $||\cdot|| : V \to \mathbb{R}$ such that for all $\mathbf{x}$, $\mathbf{y} \in V$ and (either real or complex) scalars a:

$$||\mathbf{x}|| \geq 0 \quad \text{(Positivity)}$$
$$||\mathbf{x}|| = 0 \Leftrightarrow \mathbf{x} = \mathbf{0} \quad \text{(Definiteness)}$$
$$||a\mathbf{x}|| = |a| \cdot ||\mathbf{x}|| \quad \text{(Factoring Scalars)}$$
$$||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}|| \quad \text{(Triangle Inequality)}$$

Norms define metrics (the *induced* metric):

$$d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}|| \tag{4}$$

The Euclidean metric on $\mathbb{R}^N$ comes from the Euclidean norm on $\mathbb{R}^N$, which is the usual Pythagorean length of vectors in N-space. ■

Example 1.5 An inner product on a (real) vector space V is a function $\langle\cdot|\cdot\rangle : V \times V \to \mathbb{R}$ such that for all $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z} \in V$ and $a, b \in \mathbb{R}$:

$$\langle\mathbf{x}|\mathbf{y}\rangle = \langle\mathbf{y}|\mathbf{x}\rangle \quad \text{(Symmetry)}$$
$$\langle a\mathbf{x} + b\mathbf{y}|\mathbf{z}\rangle = a\langle\mathbf{x}|\mathbf{z}\rangle + b\langle\mathbf{y}|\mathbf{z}\rangle \quad \text{(Linearity)}$$
$$\mathbf{x} \neq \mathbf{0} \Rightarrow \langle\mathbf{x}|\mathbf{x}\rangle > 0 \quad \text{(Positive-Definiteness)}$$

Inner-products define norms (the *induced* norm):

$$||\mathbf{x}|| = \sqrt{\langle\mathbf{x}|\mathbf{x}\rangle} \tag{5}$$

The Euclidean norm on $\mathbb{R}^N$ is the norm induced by the Euclidean dot product:

$$\langle\mathbf{x}|\mathbf{y}\rangle = \mathbf{x} \cdot \mathbf{y} = \sum_{n=0}^{N-1} x_n y_n \tag{6}$$

The metric induced by the induced norm is the metric *induced* by the inner product. ■

Topological spaces generalize metric spaces by axiomatizing the properties of *open* subsets. In a metric space this is very pictorial.

Definition 1.2 (Open Subsets (Metric Space)) An open subset in a metric (X, d) is a subset $\mathcal{U} \subseteq X$ such that for all $x \in \mathcal{U}$ the is an $\varepsilon > 0$ such that for all $y \in \mathcal{U}$ with $d(x, y) < \varepsilon$ it is true that $y \in \mathcal{U}$. That is, the *open ball* of radius ε centered about x fits entirely inside $\mathcal{U}$ (Fig. 1). ■

Theorem 1.1. *The collection τ of all open subsets in a metric space (X, d) satisfies the following:*

$$\emptyset \in \tau \quad \text{(The Empty Set is Open)}$$
$$X \in \tau \quad \text{(The Entire Space is Open)}$$
$$\mathcal{O} \subseteq \tau \Rightarrow \bigcup \mathcal{O} \in \tau \quad \text{(The Union of Open Sets is Open)}$$
$$\mathcal{U}, \mathcal{V} \in \tau \Rightarrow \mathcal{U} \cap \mathcal{V} \in \tau \quad \text{(The Intersection of Two Open Sets is Open)}$$

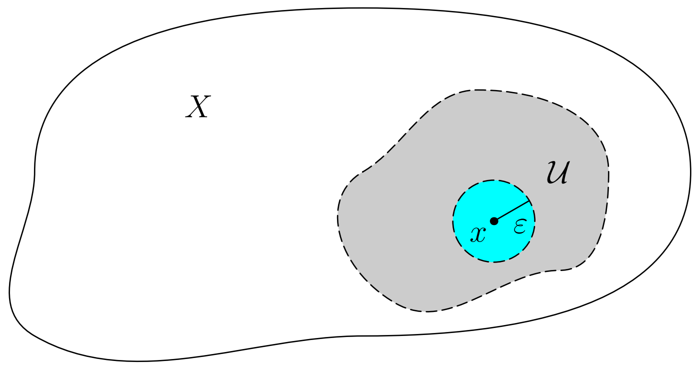

Figure 1: Open Subset in a Metric Space

The definition of a topological space is a mimicry of this theorem.

Definition 1.3 (Topological Space) A topological space is an ordered pair (X, τ) such that X is a set and $\tau \subseteq \mathcal{P}(X)$ (the power set of X) is such that:

$$\emptyset \in \tau \quad \text{(The Empty Set is Open)}$$
$$X \in \tau \quad \text{(The Entire Space is Open)}$$
$$\mathcal{O} \subseteq \tau \Rightarrow \bigcup \mathcal{O} \in \tau \quad \text{(The Union of Open Sets is Open)}$$
$$\mathcal{U}, \mathcal{V} \in \tau \Rightarrow \mathcal{U} \cap \mathcal{V} \in \tau \quad \text{(The Intersection of Two Open Sets is Open)}$$

The sets $\mathcal{U} \in \tau$ are called *open* and τ is called a *topology* on X. ■

Closed sets are the complement of open sets.

Example 1.6 If X is any set then $\{\emptyset, X\}$ is a topology on X. This is the *chaotic* or *indiscrete* topology, also called the *trivial* topology. It states that the only open subsets of X are the empty set and the whole space. ■

Example 1.7 If X is any set, then $\mathcal{P}(X)$ is a topology on X. This is the *discrete* topology. It states that *every* subset is open. ■

Metrizable spaces are those where the topology is induced by a metric. Not every topological space is induced by a metric. The trivial topology on a set with at least two distinct points is an example of a non-metrizable topological space. We can prove this by noting there is a topological property that this space lacks that all metric spaces have. The easiest such property is the *Hausdorff* one.

Definition 1.4 (Hausdorff Topological Space) A Hausdorff topological space is a topological space (X, τ) such that for all distinct $x, y \in X$ there are open sets $\mathcal{U}, \mathcal{V} \in \tau$ such that $x \in \mathcal{U}$, $y \in \mathcal{V}$, and $\mathcal{U} \cap \mathcal{V} = \emptyset$ (Fig. 2). ■

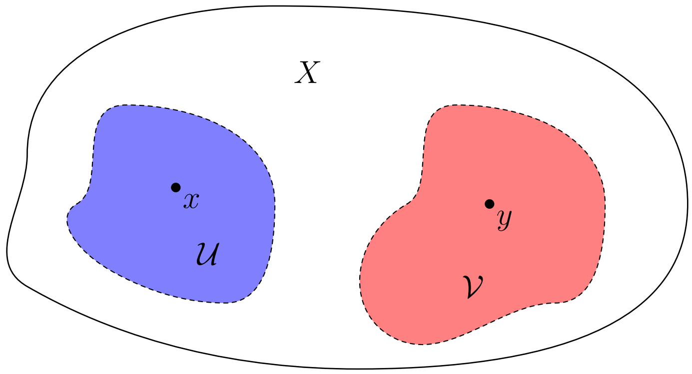

Figure 2: The Hausdorff Property

Theorem 1.2. *Metrizable spaces are Hausdorff.*

Proof. Given a metric space $(X,\, d)$ with distinct $x,\, y \in X$, we have $d(x,\, y) > 0$. Let $\varepsilon = \frac{d(x,\, y)}{2}$, and $\mathcal{U}$ and $\mathcal{V}$ be the ε balls about x and y, respectively. Then $x \in \mathcal{U}$, $y \in \mathcal{V}$, and from the triangle inequality we get $\mathcal{U} \cap \mathcal{V} = \emptyset$. □

Given the trivial topology on a set with at least two distinct points $x, y \in X$, we see that this space is not Hausdorff since these points can not be separated by open sets. Hence this space is not induced by a metric.

1.1 Subspaces

Given a topological space $(X,\, \tau)$ and a subset $A \subseteq X$ we can get a new space by considering the *subspace topology.*

Definition 1.5 (Subspace Topology) The subspace topology of a subset $A \subseteq X$ with respect to a topological space $(X,\, \tau)$ is the set τ_A defined by:

$$\tau_A = \{\, A \cap \mathcal{U} \mid \mathcal{U} \in \tau \,\} \tag{7}$$

That is, the set of all intersections of A with the open subsets of X. ■

Theorem 1.3. *The subspace topology is indeed a topology.*

Proof. Given a topological space $(X,\, \tau)$ and $A \subseteq X$, we have $A = A \cap X$, and since $X \in \tau$, it is true that $A \in \tau_A$. Similarly since $\emptyset \in \tau$ and $\emptyset = A \cap \emptyset$, we obtain $\emptyset \in \tau_A$. For unions we invoke the distributive law. A collection of open subsets of τ_A are of the form $\mathcal{U} \cap A$. Taking their union we get:

$$= \bigcup_{\mathcal{U}} (\mathcal{U} \cap A) = \Big(\bigcup_{\mathcal{U}} \mathcal{U} \Big) \cap A \tag{8}$$

since τ is a topology, $\bigcup_{\mathcal{U}} \mathcal{U}$ is open in τ, and hence this final set is open in τ_A. Lastly, given $\mathcal{U} \cap A$ and $\mathcal{V} \cap A$, we have:

$$(\mathcal{U} \cap A) \cap (\mathcal{V} \cap A) = (\mathcal{U} \cap \mathcal{V}) \cap A \tag{9}$$

Since τ is a topology, $\mathcal{U} \cap \mathcal{V}$ is open in τ, and hence $(\mathcal{U} \cap \mathcal{V}) \cap A$ is open in τ_A. So τ_A is a topology on A. □

Example 1.8 The unit N-sphere $\mathbb{S}^N$ is the subset of $\mathbb{R}^{N+1}$ defined by:

$$\mathbb{S}^N = \{\, \mathbf{x} \in \mathbb{R}^{N+1} \mid ||\mathbf{x}|| = 1 \,\} \tag{10}$$

That is, the set of all points of unit length. The standard topology is the subspace topology induced by the standard topology on $\mathbb{R}^{N+1}$, which is induced by the Euclidean metric. ■

2 Continuity

From calculus we know how to describe continuity. Minor perturbations in the domain result in small changes in the range. To be precise, given a function $f : \mathbb{R} \to \mathbb{R}$ and a point $x_0 \in \mathbb{R}$, we'll claim that f is continuous here if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $|x - x_0| < \delta$ we have $|f(x) - f(x_0)| < \varepsilon$.

This definition is adapted to metric spaces immediately.

Definition 2.1 (Continuous Function (Metric Space)) A continuous function from a metric space $(X,\, d_X)$ to a metric space $(Y,\, d_Y)$ is a function $f : X \to Y$ such that for all $x_0 \in X$ and for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x \in X$ with $d_X(x,\, x_0) < \delta$ we have $d_Y\big(f(x),\, f(x_0)\big) < \varepsilon$. ■

This makes use of real numbers and metrics, neither of which are available in the general topological setting. We instead use open sets to define continuity. This is motivated by the following.

Theorem 2.1. *If $(X,\, d_X)$ and $(Y,\, d_Y)$ are metric spaces, and if $f : X \to Y$ is a function, then f is continuous if and only if for all open $\mathcal{V} \subseteq Y$, the pre-image $f^{-1}[\mathcal{V}] \subseteq X$ is open.*

Proof. Suppose $\mathcal{V} \subseteq Y$ is open and f is continuous. If $f^{-1}[\mathcal{V}] = \emptyset$ we are done since the empty set is open. If not, let $x \in f^{-1}[\mathcal{V}]$. Since $\mathcal{V}$ is open, there is an $\varepsilon > 0$ such that the ε ball about $y = f(x)$ is contained inside $\mathcal{V}$. But f is continuous, so there is a $\delta > 0$ such that for all $x_0 \in X$ with $d_X(x,\, x_0) < \delta$ we have $d_Y\big(f(x),\, f(x_0)\big) < \varepsilon$. But this implies $x_0 \in f^{-1}[\mathcal{V}]$. That is, the δ ball about x is a subset of $f^{-1}[\mathcal{V}]$, and hence this set is open.

In the other direction, let $x \in X$ and $\varepsilon > 0$ be given. Let $\mathcal{V}$ be the ϵ ball about $y = f(x)$. Since this is open, $f^{-1}[\mathcal{V}]$ is open. But then there is a $\delta > 0$ such that the δ ball about x is contained within $f^{-1}[\mathcal{V}]$. But then for all $x_0 \in X$ such that $d_X(x,\, x_0) < \delta$, we have $d_Y\big(f(x),\, f(x_0)\big) < \varepsilon$. Hence f is continuous. □

Topological spaces do have a notion of open sets, meaning we can take this theorem and turn it into a definition.

Definition 2.2 (Continuous Function (Topological Space)) A continuous function from a topological space $(X,\, \tau_X)$ to a topological space $(Y,\, \tau_Y)$ is a function $f : X \to Y$ such that for all $\mathcal{V} \in \tau_Y$ it is true that $f^{-1}[\mathcal{V}] \in \tau_X$. ■

Example 2.1 If $(X,\, \tau)$ is any topological space, if $(Y,\, \tau_Y)$ is the indiscrete topological space on Y ($\tau_Y = \{\,\emptyset,\, Y\,\}$), and if $f : X \to Y$ is *any* function, then f is continuous. The only open sets to check are $\emptyset$ and Y. But $f^{-1}[\emptyset] = \emptyset$, which is open, and $f^{-1}[Y] = X$, which is also open. So f is continuous. ■

Example 2.2 If $(Y,\, \tau)$ is any topological space, if $(X,\, \mathcal{P}(X))$ is the discrete topological space on X, and if $f : X \to Y$ is *any* function, then f is continuous. Regardless of the open sets $\mathcal{V} \in \tau$, we have $f^{-1}[\mathcal{V}] \subseteq X \in \mathcal{P}(X)$, which is open, so f is continuous. ■

2.1 Category Theory

A small rant, I'm not a fan of category theory. Any subject that requires *proper classes* is, to me, a fiction. Nevertheless the language can at times be helpful, pedagogically. The discrete and indiscrete topologies are such examples where this language can be useful. A category $\mathbf{C}$ is a *thing* (almost never a set, maybe not even class, pending on who you ask) consisting of:

- A class $\mathrm{obj}(\mathbf{C})$ (perhaps proper) of *objects*.
- A class $\mathrm{hom}(\mathbf{C})$ (again, perhaps proper) of *arrows* between objects.
- A class function $\mathrm{dom} : \mathrm{hom}(\mathbf{C}) \to \mathrm{obj}(\mathbf{C})$ called the *domain*.
- A class function $\mathrm{cod} : \mathrm{hom}(\mathbf{C}) \to \mathrm{obj}(\mathbf{C})$ called the *codomain*.
- A *composition operator* $\mathrm{hom}(A,\, B) \times \mathrm{hom}(B,\, C) \to \mathrm{hom}(A,\, C)$ for all three objects A, B, C. Here $\mathrm{hom}(A,\, B)$ denotes the subclass of $\mathrm{hom}(\mathbf{C})$ of arrows f such that $\mathrm{dom}(f) = A$ and $\mathrm{cod}(f) = B$.
- Associativity holds: $(f \circ g) \circ h = f \circ (g \circ h)$ where $\circ$ is the composition operator.
- For all objects X there is an identity arrow $\mathrm{id}_X : X \to X$ such that for every arrow f with $\mathrm{dom}(f) = X$ we have $f \circ \mathrm{id}_X = f$ and for every arrow g with $\mathrm{cod}(f) = X$ we have $\mathrm{id}_X \circ g = g$.

Given two categories $\mathbf{C}$ and $\mathbf{D}$ a *functor* F is a *thing* (often called a mapping, but not in the sense of set theory) such that:

- For every object X in $\mathrm{obj}(\mathbf{C})$ there is an object $F(X)$ in $\mathrm{obj}(\mathbf{D})$.
- For every arrow f in $\mathrm{hom}(\mathbf{C})$ there is an arrow $F(f)$ in $\mathrm{hom}(\mathbf{D})$ such that $\mathrm{dom}(F(f)) = F(\mathrm{dom}(f))$ and $\mathrm{cod}(F(f)) = F(\mathrm{cod}(f))$.
- For each X in $\mathrm{obj}(\mathbf{C})$ the identity arrow is preserved, $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.

- For all f and g in $\hom(\mathbf{C})$, composition is preserved, $F(f \circ g) = F(f) \circ F(g)$.

A *small* category is a category $\mathbf{C}$ where $\text{obj}(\mathbf{C})$ and $\hom(\mathbf{C})$ are sets (not proper classes). The study of small categories can be done in ZFC entirely without proper classes. Groupoids, which appear in topology, geometry, and analysis, are small categories in which all arrows are invertible (have reverse arrows). *Locally small* categories are categories $\mathbf{C}$ where for all objects A, B in $\text{obj}(\mathbf{C})$ the subclass $\hom(A, B)$ is a set.

Example 2.3 The category **Set** has as objects the (proper class) of all sets.[1] The *arrows* are just functions. Every set X has an identity function $\text{id}_X : X \to X$, and the composition of functions is associative. **Set** is locally small. Given two sets A, B, the set of all functions $\mathcal{F}(A, B)$ from A to B is provably a set within the framework of ZFC.[2] ■

Example 2.4 The category **Top** has as objects the (proper class)[3] of all topological spaces. The arrows are continuous functions. This category is also locally small, given (X, τ_X) and (Y, τ_Y), the collection $C\big((X, \tau_X), (Y, \tau_Y)\big)$ of all continuous functions $f : X \to Y$ is a set, being a subset of the set $\mathcal{F}(X, Y)$ of all functions $f : X \to Y$. ■

Example 2.5 The category **Grp** has as objects the (proper class)[4] of all groups. The arrows are group homomorphisms. **Grp** is locally small since the collection of all group homomorphisms $\varphi : G \to H$ is a subset of $\mathcal{F}(G, H)$. ■

I know of no examples of categories that are not locally small. After some digging I found the category of *spans*, but I don't know what these are. Some claim there is a category **Cat** of all categories, but that just sounds like Russell's paradox waiting to happen.

In a locally small category, the subclasses $\hom(A, B)$ are sets, so we call them *homsets*. Given two locally small categories $\mathbf{C}$ and $\mathbf{D}$, two objects X, Y in $\text{obj}(\mathbf{C})$, and a functor $F : \mathbf{C} \to \mathbf{D}$, we get a function (an actual function from set theory) $F_{X,Y} : \hom(X, Y) \to \hom\big(F(X), F(X)\big)$. F is called *faithful* if $F_{X,Y}$ is injective for all objects X and Y. It is called *full* if $F_{X,Y}$ is surjective for all objects X and Y.

Example 2.6 The functor $F : \mathbf{Top} \to \mathbf{Set}$ defined by $F\big((X, \tau)\big) = X$ for objects and $F(f) = f$ for arrows (a continuous function is a function, after all) is faithful. Given two topological spaces (X, τ_X) and (Y, τ_Y) the function $F_{X,Y} : C\big((X, \tau_X), (Y, \tau_Y)\big) \to \mathcal{F}(X, Y)$, which is $F_{X,Y}(f) = f$, is injective, but in general it is not surjective. It is not surjective since there may be functions $f : X \to Y$ that are not continuous. F is called the *forgetful* functor. It can be similarly defined for **Grp**. ■

[1] See Russell's paradox for why this is not a set.

[2] Using the ordered pair definition of function, $\mathcal{F}(A, B)$ is a subset of $\mathcal{P}\big(\mathcal{P}(A \times B)\big)$.

[3] Every set has a corresponding topological space, the discrete topology. Intuitively there are as many sets as there are topological spaces, so the collection of all topological spaces is not a set.

[4] For every set, there is a group. This is equivalent to axiom of choice.

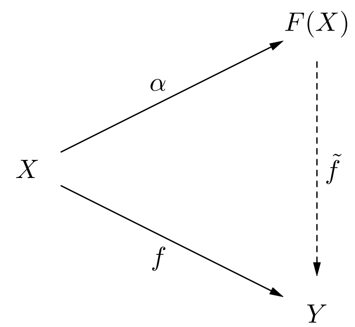

Figure 3: Free Object in Topology

A concrete category is a locally small category $\mathbf{C}$ with a faithful functor $U : \mathbf{C} \to \mathbf{Set}$. **Top** and **Grp** with the forgetful functor form concrete categories. A *free object* from a set X in a concrete category $(\mathbf{C}, U)$ is an object $F(X)$ in $\text{obj}(\mathbf{C})$ with an injective function $\alpha : X \to U\big(F(X)\big)$ with the following property. Given any object Y in $\text{obj}(\mathbf{C})$ and any function $f : X \to U(Y)$ there is a unique arrow $\tilde{f} : F(X) \to Y$ such that $\tilde{f} = U(f) \circ \alpha$. In other words, the diagram in Fig. 3 is commutative.

Let's rephrase this in the language of topology. We have a set X and we want a topological space $F(X)$ with an injective function $\alpha : X \to U\big(F(X)\big)$ such that for *any* topological space (Y, τ_Y) and *any* function $f : X \to Y$ there is a unique continuous function $\tilde{f} : F(X) \to (Y, \tau_Y)$ such that $f = U(\tilde{f}) \circ \alpha$.

We've seen this already. Define $U\big((X, \tau)\big) = X$ (the forgetful functor), $F(X) = \big(X, \mathcal{P}(X)\big)$ (the discrete topology), and let $\alpha : X \to X$ be the identity $\alpha = \text{id}_X$. Given any topological space (Y, τ_Y) and any function $f : X \to Y$, the unique continuous function is $\tilde{f} = f$.

To summarize, the discrete topology is the *free object* in the category **Top**. Thus the discrete topology is analogous to the free group in algebra. The underlying set X acts as a *basis* for the topological space, just like generators act as a basis for free group. This is also similar to bases which are used to generate vector spaces.

The *cofree* object flips all the arrows. Given a set X we want a topological space $C(X)$ and an injective function $\alpha : X \to U\big(C(X)\big)$ such that for any space (Y, τ_Y) and any function $f : Y \to X$, there is a unique continuous function $\tilde{f} : (Y, \tau_Y) \to C(X)$ such that $f = \alpha \circ U(\tilde{f})$.

We've seen this too. Let U be the forgetful functor, and define $C(X) = (X, \{\emptyset, X\})$. The injective function $\alpha : X \to X$ is once again the identity, $\alpha = \text{id}_X$. Then given any topological space (Y, τ_Y) and any function $f : Y \to X$,

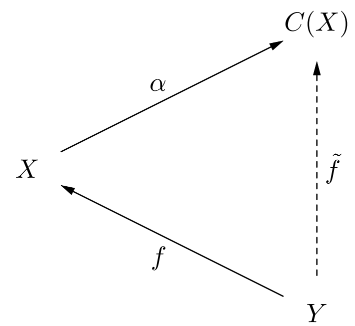

Figure 4: Cofree Object in Topology

the unique continuous function that does the trick is $\tilde{f} = f$ (See Fig. 4). That is, the cofree object in **Top** is the indiscrete topology.

2.2 Homeomorphisms

The arrows in category theory are called *morphisms*. Given a category $\mathbf{C}$, two objects A, B in obj($\mathbf{C}$), and an arrow $f : A \to B$, an inverse is an arrow $g : B \to A$ such that the composition operations $f \circ g$ and $g \circ f$ yield the identity morphism. *Isomorphisms* are morphisms that have inverses.

Example 2.7 In **Set** the morphisms are just functions. The invertible functions are precisely those that are bijective. That is, the isomorphisms in **Set** are bijections. ■

Example 2.8 In **Grp** the morphisms are group homomorphisms. A bijective group homorphism automatically yields a group homomorphism for the inverse. The isomorphisms in **Grp** are bijective group homomorphisms, also called group isomorphisms. ■

Example 2.9 In $\mathbf{Vec}_{\mathbb{R}}$, the category of real vector spaces, the morphisms are linear transformations. Bijective linear transformations have linear inverses, so the isomorphisms in $\mathbf{Vec}_{\mathbb{R}}$ are just bijective linear transformations. ■

In topology the morphisms are continuous functions. Unlike the aforementioned algebraic structures, bijective continuous functions need not yield continuous inverses.

Example 2.10 Consider the circle $\mathbb{S}^1$ and half-open interval $[0,\, 1)$, both with their standard subspace topologies. The function $\varphi : [0,\, 1) \to \mathbb{S}^1$ defined by:

$$\varphi(t) = \big(\cos(2\pi t),\, \sin(2\pi t)\big) \tag{11}$$

is continuous (since it is continuous in both components) and bijective, but the inverse is not continuous. The inverse function creates a *rip* at the point $(1,\, 0)$ (you can prove this using a $\varepsilon - \delta$ argument since these are metric spaces). ■

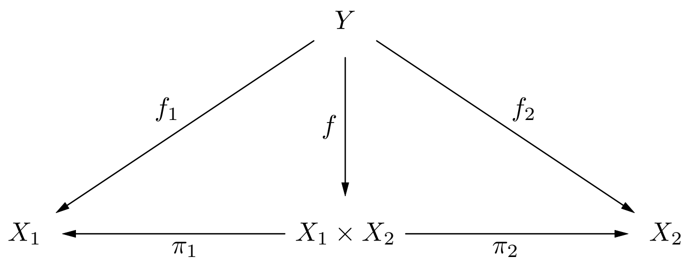

Figure 5: Categorical Diagram for Products in Topology

The isomorphisms in topology are continuous bijective functions with continuous inverses. These are given a name.

Definition 2.3 (Homeomorphism) A homeomorphism from a topological space (X, τ_X) to a topological space (Y, τ_Y) is a continuous bijective function $f : X \to Y$ such that $f^{-1} : Y \to X$ is continuous. ■

Example 2.11 Equip $\mathbb{R}$ with the Euclidean topology. The function $f(x) = x^3$ is continuous and bijective, and the inverse $f^{-1}(x) = x^{1/3}$ is continuous as well, meaning f is a homeomorphism. A homeomorphism from a space to itself is sometimes called an *autohomeomorphism.* ■

Definition 2.4 (Open Mapping) An open mapping from a topological space (X, τ_X) to a topological space (Y, τ_Y) is a function $f : X \to Y$ such that for all $\mathcal{U} \in \tau_X$ it is true that $f[\mathcal{U}] \in \tau_Y$. ■

Open mappings need not be continuous, and continuous functions do not need to be open mappings. When we do have both, we're not far from a homeomoprhism.

Theorem 2.2. *If (X, τ_X) and (Y, τ_Y) are topological spaces, and if $f : X \to Y$ is a function, then f is a homeomorphism if and only if it is a continuous bijective open mapping.*

2.3 Product Spaces

We need to discuss product spaces in order to talk about homotopy. Products in topology are defined in one of two ways. The first, and perhaps more natural, way is by *generating* the topology. Given two spaces (X, τ_X) and (Y, τ_Y), we form the product space $(X \times Y, \tau_{X\times Y})$ by looking at the products of open sets in τ_X and τ_Y. The collection of all such sets is not usually a topology, so we define $\tau_{X\times Y}$ to be the *smallest* topology such that $\mathcal{U} \times \mathcal{V} \in \tau_{X\times Y}$ for all $\mathcal{U} \in \tau_X$ and $\mathcal{V} \in \tau_Y$.

Definition 2.5 (Product Topological Space) The product of two topological spaces (X, τ_X) and (Y, τ_Y) is the space $(X \times Y, \tau_{X\times Y})$ where $\tau_{X\times Y}$ is the

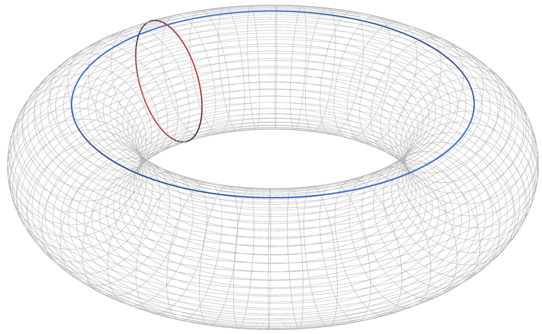

Figure 6: Torus as a Product of Circles

smallest topology that contains the set:

$$\tilde{\tau}_{X\times Y} = \{\,\mathcal{U}\times\mathcal{V}\mid \mathcal{U}\in\tau_X \text{ and } \mathcal{V}\in\tau_Y\,\} \tag{12}$$

That is, the topology *generated* by the product of open sets. ■

This, to me, is the more intuitive definition. It generalizes to finite products by induction. Infinite products have the issue of choosing the *product* or the *box* topologies, see my notes for Math 54. We'll be mostly concerned with finite products in this course.

The alternate definition is categorical. The use is that it has the same definition for the product of groups, vector spaces, sets, etc. For the product of sets we have the canonical projections $\pi_1 : X_1 \times X_2 \to X_1$ defined by $\pi_1(x_1,\, x_2) = x_1$, and similarly for π_2. The product topological space is defined by the object in **Top** with the following property. Given *any* topological space $(Y,\, \tau_Y)$, and any two continuous functions $f_1 : Y \to X_1$ and $f_2 : Y \to X_2$, there is a unique continuous function $f : Y \to X_1 \times X_2$ that makes Fig. 5 commute.

This says that to check the continuity of $f : Y \to X_1 \times X_2$ it is sufficient to check that the components of f are continuous.

Example 2.12 The function $f : \mathbb{R} \to \mathbb{R}^3$ defined by $f(t) = (t,\, t^2\exp(t),\, \cos(t^2))$ is continuous. Why? We will not be looking at the pre-image of open sets, and even $\varepsilon - \delta$ proofs look tedious here. But each of the components are continuous, so we automatically know that function itself is continuous. ■

One way of thinking of product spaces is by taking a copy of X_1 and attaching it to every point of X_2 (or vice-versa). This is most easily visualized with the torus, $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$ (See Fig. 6).

2.4 Homotopy and Homotopy Equivalence

A notion weaker than homeomorphism, but equally useful, is homotopy equivalence. It is defined in terms of homotopy, which is the idea of stretching

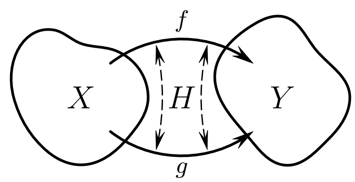

Figure 7: Pictorial Representation of Homotopy

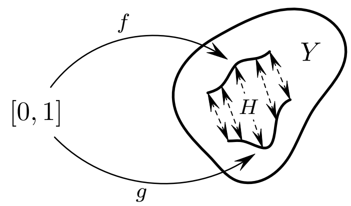

Figure 8: Straight-Line Homotopy

continuous functions in a topological space.

Definition 2.6 (Homotopy) A homotopy between continuous function $f, g : X \to Y$ from a topological space (X, τ_X) to a topological space (Y, τ_Y) is a continuous functions $H : X \times [0,\, 1] :\to Y$ (with respect to the subspace topology on $[0,\, 1]$ and the product topology on $X \times [0,\, 1]$) such that $H(x,\, 0) = f(x)$ and $H(x,\, 1) = g(x)$ for all $x \in X$. ■

This is shown pictorially in Fig. 7.

Example 2.13 Given any two continuous functions $f, g : \mathbb{R}^M \to \mathbb{R}^N$, the *straight-line homotopy* is the function $H : \mathbb{R}^M \times [0,\, 1] \to \mathbb{R}^N$ defined by:

$$H(\mathbf{x},\, t) = (1 - t)\, f(\mathbf{x}) + t\, g(\mathbf{x}) \tag{13}$$

This is continuous, being the sum of continuous functions. Plugging in $t = 0$ we get $H(\mathbf{x},\, 0) = f(\mathbf{x})$, and $t = 1$ yields $H(\mathbf{x},\, 1) = g(\mathbf{x})$. So f and g are homotopic, and H is such a homotopy. ■

The straight line homotopy is shown in Fig. 8 for two curves in a subspace $Y \subseteq \mathbb{R}^2$.

Homotopy is used to define *homotopy equivalence* which is a weaker form of equivalence for topological spaces. It is defined using *homotopy inverses*.

Definition 2.7 (Homotopy Inverse) A homotopy inverse of a continuous function $f : X \to Y$ from a topological space (X, τ_X) to a space (Y, τ_Y) is a continuous function $g : Y \to X$ such that $g \circ f$ is homotopic to id_X and $f \circ g$ is homotopic to id_X. ■

Definition 2.8 (Homotopy Equivalence) A homotopy equivalence from a topological space $(X,\, \tau_X)$ to a topological space $(Y,\, \tau_Y)$ is a continuous function $f : X \to Y$ such that there exists a homotopy inverse $g : Y \to X$ for f. ∎

Theorem 2.3. *A homeomorphism is a homotopy equivalence.*

Proof. A homeomorphism $f : X \to Y$ has the property that f^{-1} is continuous and hence $f \circ f^{-1}$ is *equal* to id_X, not just homotopy equivalent. Similarly, $f^{-1} \circ f = \mathrm{id}_Y$. So f is a homotopy equivalence. □

This theorem does not reverse.

Example 2.14 $\mathbb{R}^N$ is homotopy equivalent to a one-point space $\{\, 0 \,\}$ (There is only one topology on a one-point space, $\tau = \big\{\, \emptyset,\, \{\, 0 \,\} \,\big\}$). The homotopy equivalence $f : \mathbb{R}^N \to \{\, 0 \,\}$ is the only function that exists, $f(\mathbf{x}) = 0$. The homotopy inverse $g : \{\, 0 \,\} \to \mathbb{R}^N$ can be any function you like, but let's pick $g(0) = \mathbf{0}$ to make things simple. The composition $f \circ g$ is the identity on $\{\, 0 \,\}$, so it is certainly homotopic to the identity. Going the other way, $g \circ f$ is homotopic to the identity on $\mathbb{R}^N$. Define H by:

$$H(\mathbf{x},\, t) = t\mathbf{x} \tag{14}$$

$H(\mathbf{x},\, 0)$ is the function $g \circ f$, and $H(\mathbf{x},\, 1)$ is the identity. This is the *straight-line* homotopy we've seen before. ∎

$\mathbb{R}^N$ is an example of a *contractible* space.

Definition 2.9 (Contractible Topological Space) A contractible topological space is a space $(X,\, \tau)$ that is homotopy equivalent to a one-point space. ∎

We saw that every continuous function from $\mathbb{R}^M$ to $\mathbb{R}^N$ is homotopy equivalent via the straight-line homotopy. The real culprit behind this is contractibility.

Theorem 2.4. *If $(X,\, \tau_X)$ is a topological space, if $(Y,\, \tau_Y)$ is contractible, and if $f, g : X \to Y$ are continuous, then they are homotopy equivalent.*

Proof. We can exclude the case of X or Y being empty as trivial. Let $y \in Y$. There is only one one-point space, so $\{\, y \,\}$ with the subspace topology is a one point space. Since $(Y,\, \tau_Y)$ is contractible, there is a homotopy equivalence $\alpha : Y \to \{\, y \,\}$ and a homotopy inverse $\beta : \{\, y \,\} \to Y$. Let H be a homotopy between id_Y and $\beta \circ \alpha$. Then $G : X \times [0,\, 1] \to Y$ defined by:

$$G(x,\, t) = \begin{cases} H\big(f(x),\, 2t\big), & 0 \leq t \leq \frac{1}{2} \\ H\big(g(x),\, 2 - 2t\big), & \frac{1}{2} \leq t \leq 1 \end{cases} \tag{15}$$

is continuous by the pasting-lemma since $H\big(f(x),\, 1\big) = H\big(g(x),\, 1\big) = \beta(y)$. G is thus a homotopy between f and g. □

You can get a different category out of topology. The objects are still topological spaces, but the arrows are *equivalent classes* of continuous functions under the equivalence relation of homotopic. Isomorphisms are then equivalence classes of homotopy equivalences. Two topological spaces are then considered the *same* if they are homotopy equivalent. This allows a lot more squooshing of the space ($\mathbb{R}^N$ is homotopy equivalent to a single point but certainly not homeomorphic).

3 Compactness and Connectedness

I think that was more than enough category theory for a while. Let's return to simpler point-set language and review some more topological ideas.

3.1 Compact Spaces

Definition 3.1 (Compact Topological Space) A compact topological space is a topological space (X, τ) such that for every open over (a set $\mathcal{O} \subseteq \tau$ such that $\bigcup \mathcal{O} = X$) there is a finite subset $\Delta \subseteq \mathcal{O}$ that covers X. ■

Theorem 3.1 (Heine-Borel Theorem)**.** *A subset $\mathcal{C} \subseteq \mathbb{R}^N$ is compact if and only if $\mathcal{C}$ is closed and bounded (with respect to the Euclidean metric).*

Theorem 3.2 (Generalized Heine-Borel Theorem)**.** *A subset $\mathcal{C} \subseteq X$ of a metric space is compact if and only if it is closed and totally-bounded.*

Both of these theorems were proved in detail in Math 54. We'll make frequent use of them.

3.2 Connected Spaces

There are several notions of a space being *connected*, and in a course on planar topology it is essential to note the differences. The simplest notion of connectedness uses open sets and describes how to *disconnect* a space.

Definition 3.2 (Disconnected Topological Space) A disconnected topological space is a topological space (X, τ) such that there exist two disjoint non-empty open subsets $\mathcal{U}, \mathcal{V} \subseteq \tau$ such that $\mathcal{U} \cup \mathcal{V} = X$. ■

Definition 3.3 (Connected Topological Space) A connected topological space is a topological space that is not disconnected. ■

Stronger than connectedness is the notion of *path* connected.

Definition 3.4 (Path Connected Topological Space) A path connected topological space is a topological space (X, τ) such that for all $x, y \in X$ there is a continuous path $\gamma : [0, 1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. ■

Theorem 3.3. *If (X, τ) is a path connected topological space, then it is connected.*

Proof. Suppose not. Then there are two non-empty disjoint open subsets $\mathcal{U}, \mathcal{V} \subseteq X$ such that $\mathcal{U} \cup \mathcal{V} = X$. Since they are non-empty, there are points $x \in \mathcal{U}$ and

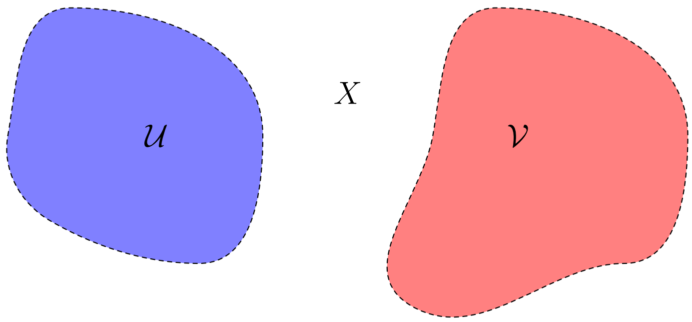

Figure 9: A Disconnected Topological Space

$y \in \mathcal{V}$. But (X, τ) is path connected, so there is a continuous path $\gamma : [0, 1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Since γ is continuous, and since $\mathcal{U}$ and $\mathcal{V}$ are open, $\gamma^{-1}[\mathcal{U}]$ and $\gamma^{-1}[\mathcal{V}]$ are open. But these subsets are also disjoint and non-empty, meaning $[0, 1]$ is disconnected, but it is not, a contradiction. So (X, τ) is connected. □

One more stronger notion.

Definition 3.5 (Arc Connected Topological Space) An arc connected topological space is a topological space (X, τ) such that for all $x, y \in X$ there is an injective continuous path $\gamma : [0, 1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. ■

Every path connected Hausdorff space is arc connected. This will take a bit of work to prove but many of the central ideas of planar topology are involved. We'll be talking a lot about arc connected spaces when we discuss space filling curves.

4 More Topological Properties

Lastly, a very brief review of some more properties covered in Math 54.

4.1 Separation Properties

Definition 4.1 (Fréchet Topological Space) A Fréchet topological space, also called a T_1 space, is a topological space (X, τ) such that for all distinct $x, y \in X$ there are open sets $\mathcal{U}, \mathcal{V} \in \tau$ such that $x \in \mathcal{U}$, $x \notin \mathcal{V}$, and $y \in \mathcal{V}$, $y \notin \mathcal{U}$ (Fig. 10). ■

Definition 4.2 (Regular Topological Space) A regular topological space is a space (X, τ) such that for all $x \in X$ and for all closed $\mathcal{C} \subseteq X$ with $x \notin \mathcal{C}$ there exist disjoint open subsets $\mathcal{U}, \mathcal{V} \in \tau$ such that $x \in \mathcal{U}$ and $\mathcal{C} \subseteq \mathcal{V}$ (Fig. 11). ■

Definition 4.3 (Normal Topological Space) A normal topological space is a space (X, τ) such that for all disjoint closed sets $\mathcal{C}, \mathcal{D} \subseteq X$ there exist disjoint open subsets $\mathcal{U}, \mathcal{V} \in \tau$ such that $\mathcal{C} \subseteq \mathcal{U}$ and $\mathcal{D} \subseteq \mathcal{V}$ (Fig. 12). ■

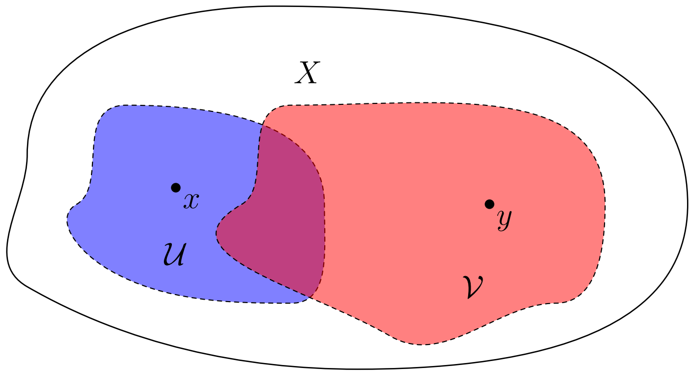

Figure 10: The Fréchet Condition for Topological Spaces

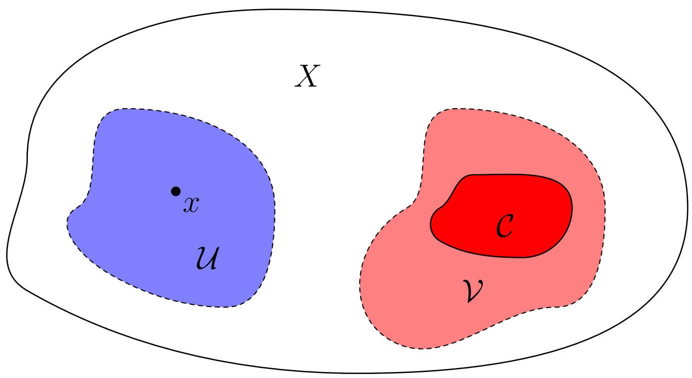

Figure 11: The Regular Condition for Topological Spaces

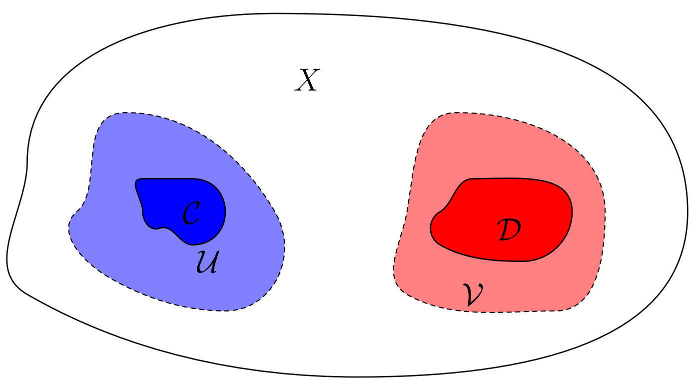

Figure 12: The Normal Condition for Topological Spaces

4.2 Countability Properties

Definition 4.4 (Base (Topology)) A base for a topology τ on a set X is a subset $\mathcal{B} \subseteq \tau$ such that $\bigcup \mathcal{B} = X$ ($\mathcal{B}$ is an open cover), and for all $\mathcal{U}, \mathcal{V} \in \mathcal{B}$, and for all $x \in \mathcal{U} \cap \mathcal{V}$, there is a $\mathcal{W} \in \mathcal{B}$ such that $x \in \mathcal{W}$ and $\mathcal{W} \subseteq \mathcal{U} \cap \mathcal{V}$. ■

Example 4.1 The set of all open intervals $(a, b) \subseteq \mathbb{R}$ form a basis for the Euclidean topology on the real line. ■

Definition 4.5 (Second Countability) A second countable topological space is a topological space (X, τ) such that there exists a countable base $\mathcal{B}$ for τ. ■

Example 4.2 The set of all open intervals (p, q) with *rational endpoint*, $p, q \in \mathbb{Q}$, forms a countable basis for the Euclidean topology on the real line. This shows that $\mathbb{R}$ is second countable. ■

Theorem 4.1 (Urysohn's Metrization Theorem). *If (X, τ) is a regular Hausdorff topological space that is second countable, then it is metrizable. That is, there is some metric d on X that induces τ.*

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